ADAPTIVE OPTIMIZED SCHWARZ METHODS

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SCHWARZ METHODS

Consider the following sample problem:

$$\Delta u(x, y) = f(x, y), \quad (x, y) \in \Omega = [-1, 1] \times [-1, 1]$$

$$u(x, y) = h(x, y), \quad (x, y) \in \partial\Omega$$

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$$\Delta u_1^n(x,y) = f(x,y), \quad (x,y) \in \Omega_1 = [-1,\alpha] \times [-1,1]$$

$$u_1^n(\alpha,y) = u_2^{n-1}(\alpha,y)$$

$$\Delta u_2^n(x,y) = f(x,y), \quad (x,y) \in \Omega_2 = [\beta,1] \times [-1,1]$$

$$u_2^n(\beta,y) = u_1^{n-1}(\beta,y)$$

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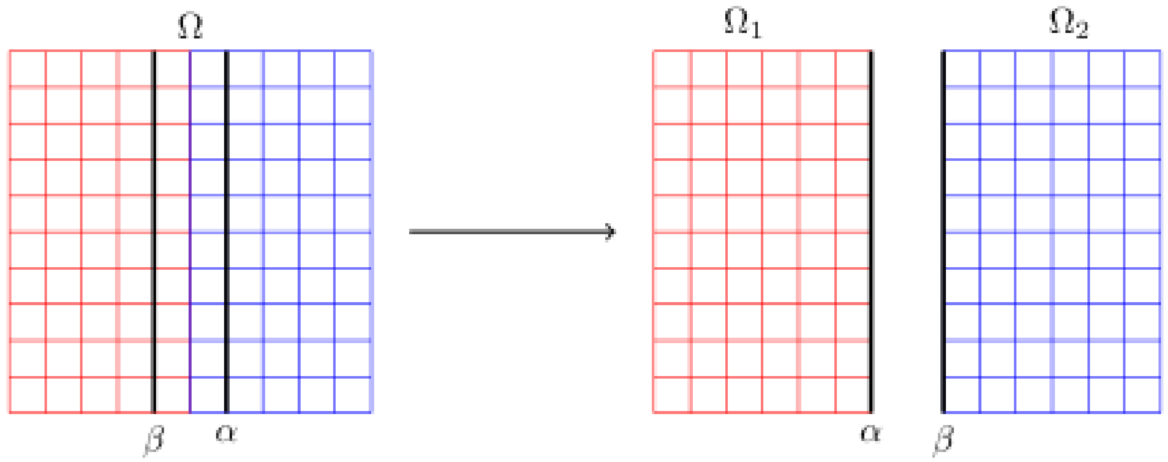

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Multiplicative Schwarz: solve one subdomain after the other

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Multiplicative Schwarz: solve one subdomain after the other

Additive Schwarz: solve both subdomains at the same time

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$$\begin{bmatrix} A_{11} & A_{1\Gamma} & \\ A_{\Gamma 1} & A_{\Gamma\Gamma} & A_{\Gamma 2} \\ & A_{2\Gamma} & A_{22} \end{bmatrix} \begin{bmatrix} \vec{u}_1 \\ \vec{u}_\Gamma \\ \vec{u}_2 \end{bmatrix} = \begin{bmatrix} \vec{f}_1 \\ \vec{f}_\Gamma \\ \vec{f}_2 \end{bmatrix}$$

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$$\begin{bmatrix} A_{11} & A_{1\Gamma} \\ A_{\Gamma 1} & A_{\Gamma\Gamma} + T_{2\to 1} \end{bmatrix} \begin{bmatrix} \vec{u}_1^{n+1} \\ \vec{u}_{1\Gamma}^{n+1} \end{bmatrix} = \begin{bmatrix} \vec{f}_1 \\ \vec{f}_\Gamma \end{bmatrix} + \begin{bmatrix} \\ -A_{\Gamma 2}\vec{u}_2^n + T_{2\to 1}\vec{u}_{2\Gamma}^n \end{bmatrix}$$

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OPTIMIZED SCHWARZ METHODS

Boundary conditions transmit information between the two subdomains

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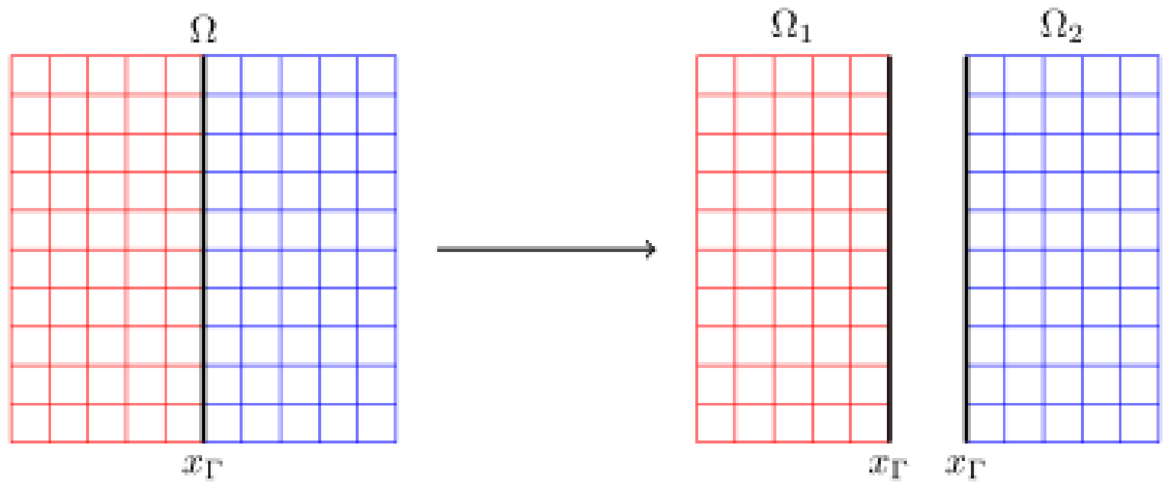

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If you have a specific Fourier mode in mind, you can also pick the best p for just that mode

Robin BCs aren't the only option for optimized BCs

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Tangential BCs are also a popular choice, and give a second parameter q to optimize

$$\frac{\partial u_1^n}{\partial x} - p u_1^n(0,y) + q \frac{\partial u_1^n}{\partial y} =$$

$$\frac{\partial u_2^{n-1}}{\partial x} - p u_2^{n-1}(0,y) + q \frac{\partial u_2^n}{\partial y}$$

But the best BCs are absorbing BCs, which are used in perfectly matched layers

These are dense, and correspond to Schur complements

$$T_{2\to 1} \to S_{2\to 1} = -A_{\Gamma 2} A_{22}^{-1} A_{2\Gamma},$$

$$T_{1\to 2} \to S_{1\to 2} = -A_{\Gamma 1} A_{11}^{-1} A_{1\Gamma}$$

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This means they have about the same computation time as M iterations, where M is the size of the overlap

ADAPTIVE TRANSMISSION CONDITIONS

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We want cheap, black box BCs

To get them, we'll find them adaptively

Let's look at the differences between iterates for a single sudomain

$$\begin{bmatrix} A_{11} & A_{1\Gamma} \\ A_{\Gamma 1} & A_{\Gamma\Gamma} + T_{2\to 1} \end{bmatrix} \left(\begin{bmatrix} \vec{u}_1^{n+1} \\ \vec{u}_{1\Gamma}^{n+1} \end{bmatrix} - \begin{bmatrix} \vec{u}_1^{n} \\ \vec{u}_{1\Gamma}^{n} \end{bmatrix} \right) =$$

$$\begin{bmatrix} \vec{f}_1 \\ \vec{f}_\Gamma \end{bmatrix} + \begin{bmatrix} \\ -A_{\Gamma 2}\vec{u}_2^n + T_{2\to 1}\vec{u}_{2\Gamma}^n \end{bmatrix} - \left(\begin{bmatrix} \vec{f}_1 \\ \vec{f}_\Gamma \end{bmatrix} + \begin{bmatrix} \\ -A_{\Gamma 2}\vec{u}_2^{n-1} + T_{2\to 1}\vec{u}_{2\Gamma}^{n-1} \end{bmatrix} \right)$$

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$$\begin{bmatrix} A_{11} & A_{1\Gamma} \\ A_{\Gamma 1} & A_{\Gamma\Gamma} + T_{2\to 1} \end{bmatrix} \begin{bmatrix} \vec{d}_1^{n+1} \\ \vec{d}_{1\Gamma}^{n+1} \end{bmatrix} = \begin{bmatrix} \\ -A_{\Gamma 2}\vec{d}_2^n + T_{2\to 1}\vec{d}_{2\Gamma}^n \end{bmatrix}$$

We then perform what's known as static condensation by noting that

$$A_{11}\vec{d}_1^{n+1} = -A_{\Gamma 1}\vec{d}_{1\Gamma}^{n+1}$$

This leads to

$$(A_{\Gamma\Gamma} + S_{1\to 2} + T_{2\to 1})\vec{d}_{1\Gamma}^{n+1} = (T_{2\to 1} - S_{2\to 1})\vec{d}_{2}^{n}$$

This leads to

$$(A_{\Gamma\Gamma} + S_{1\to2} + T_{2\to1})\vec{d}_{1\Gamma}^{\,n+1} = (T_{2\to1} - S_{2\to1})\vec{d}_{2\Gamma}^{\,n}$$

$$\vec{y}^{\,n+1} = E_2\vec{d}_{2\Gamma}^{\,n} = -A_{\Gamma 2}\vec{d}_{2}^{\,n} + T_{2\to1}\vec{d}_{2\Gamma}^{\,n}$$

$$\vec{y}^{n+1} = E_2 \vec{d}_{2\Gamma}^{n} = -A_{\Gamma 2} \vec{d}_{2}^{n} + T_{2\to 1} \vec{d}_{2\Gamma}^{n}$$

At every other iteration, we're going to update E_2

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$$E_2 \to E_2 - \frac{\vec{y}\vec{d}^{\top}}{||\vec{d}||^2}$$

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Through this process, the vectors $\vec{y}$ get modified as well, to the vectors $\vec{v}$

$$E_2 \to E_2 - VW^\top$$

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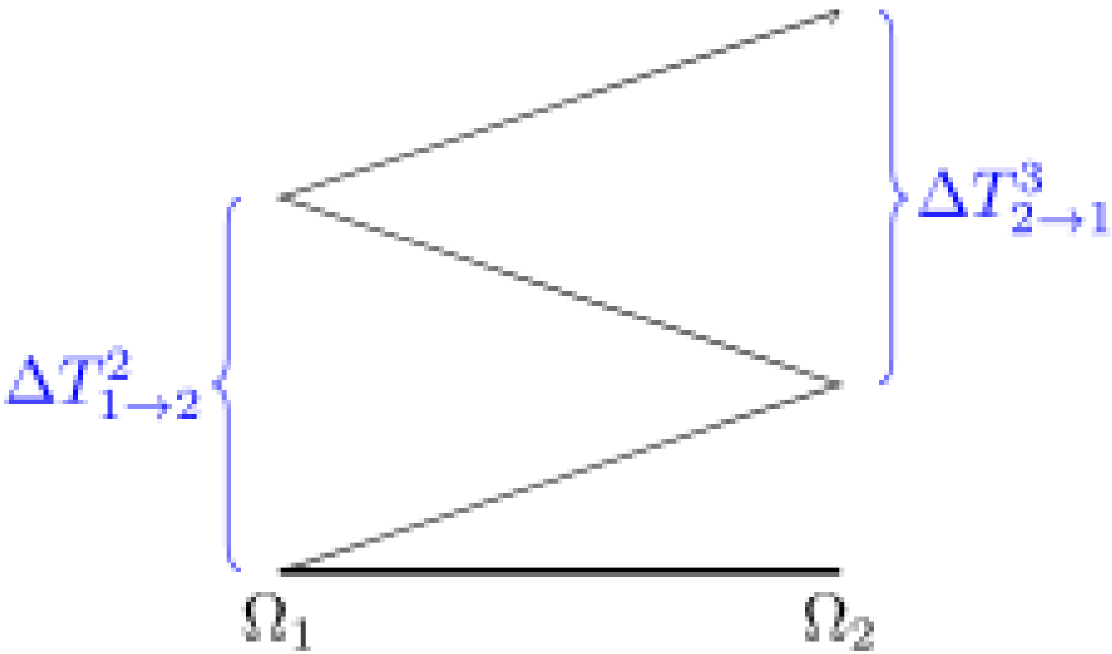
To make the optimized BCs, we subtract this from $T_{2\to1}$

The transmission conditions $T_{2\to 1}$ and $T_{1\to 2}$ now change iteratively



Recall:

$$(A_{\Gamma\Gamma} + S_{1\to 2} + T_{2\to 1})\vec{d}_{1\Gamma}^{n+1} = (T_{2\to 1} - S_{2\to 1})\vec{d}_{2\Gamma}^{n}$$

The vectors $\vec{d}$ lie in a Krylov subspace

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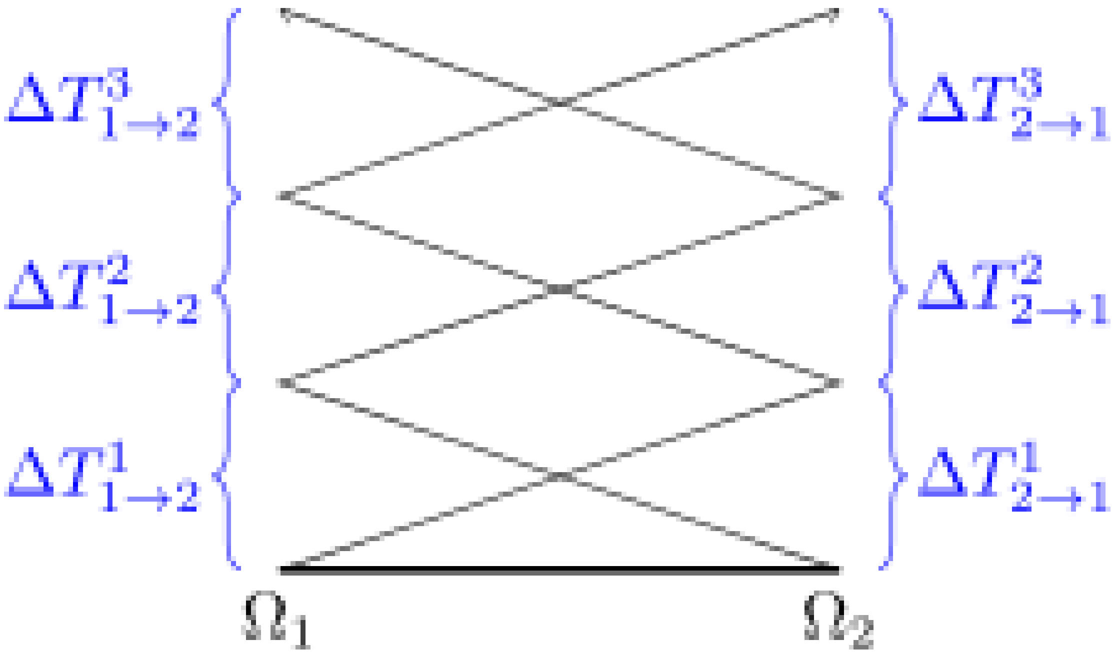
The vectors $\vec{d}$ lie in a Krylov subspace

They satisfy an implicit Galerkin condition

There's also the option to update the BCs at every iteration



ADAPTIVE OPTIMIZED SCHWARZ METHODS (AOSMS)

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Find $\vec{u}_1^0 = A_{11}^{-1}\left(\vec{f}_1 - A_{1\Gamma}\vec{u}_{1\Gamma}^0\right)$

Solve

$$\begin{bmatrix} A_{22} & A_{2\Gamma} \\ A_{\Gamma 2} & A_{\Gamma\Gamma} + T_{1\to 2}^{1} \end{bmatrix} \begin{bmatrix} \vec{u}_2^1 \\ \vec{u}_{2\Gamma}^1 \end{bmatrix} = \begin{bmatrix} \vec{f}_2 \\ \vec{f}_\Gamma \end{bmatrix} + \begin{bmatrix} \\ -A_{\Gamma 1}\vec{u}_1^0 + T_{1\to 2}^{1}\vec{u}_{1\Gamma}^0 \end{bmatrix}$$

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$$\begin{bmatrix} A_{11} & A_{1\Gamma} \\ A_{\Gamma 1} & A_{\Gamma\Gamma} + T^1_{2\to 1} \end{bmatrix} \begin{bmatrix} \vec{u}^2_1 \\ \vec{u}^2_{1\Gamma} \end{bmatrix} = \begin{bmatrix} \vec{f}_1 \\ \vec{f}_\Gamma \end{bmatrix} + \begin{bmatrix} \\ -A_{\Gamma 2}\vec{u}^1_2 + T^1_{2\to 1}\vec{u}^1_{2\Gamma} \end{bmatrix}$$

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Calculate $\vec{d}^2_{1\Gamma} = \vec{u}^2_{1\Gamma} - \vec{u}^0_{1\Gamma}$ and $\vec{d}^2_1 = \vec{u}^2_1 - \vec{u}^0_1$

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Normalize $\vec{d}^2_{1\Gamma}$ using α^2_1 such that $\vec{w}^2_1 = \alpha^2_1 \vec{d}^2_{1\Gamma}$ and calculate

$$\vec{v}^2_1 = \alpha^2_1 \left(-A_{\Gamma 1} \vec{d}^2_1 + T^1_{1\rightarrow 2} \vec{d}^2_{1\Gamma} \right)$$

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Normalize $\vec{d}_{1\Gamma}^{2}$ using α_1^2 such that $\vec{w}_1^2 = \alpha_1^2 \vec{d}_{1\Gamma}^{2}$ and calculate

$$\vec{v}_1^2 = \alpha_1^2 \left(-A_{\Gamma 1} \vec{d}_1^{2} + T_{1\to 2}^{1} \vec{d}_{1\Gamma}^{2} \right)$$

Update $T_{1\to 2}^{1}$:

$$T_{1\to 2}^{2} = T_{1\to 2}^{1} + \Delta T_{1\to 2}^{2} = T_{1\to 2}^{1} - \vec{v}_1^2 \left(\vec{w}_1^2 \right)^{\top}$$

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Solve

$$\begin{bmatrix} A_{22} & A_{2\Gamma} \\ A_{\Gamma 2} & A_{\Gamma\Gamma} + T^2_{1\to 2} \end{bmatrix} \begin{bmatrix} \vec{d}^3_2 \\ \vec{d}^3_{2\Gamma} \end{bmatrix}$$

$$= \begin{bmatrix} \\ -A_{\Gamma 1}\vec{d}^2_1 + T^2_{1\to 2}\vec{d}^2_{1\Gamma} - \Delta T^2_{1\to 2}\left(\vec{u}^1_{2\Gamma} - \vec{u}^0_{1\Gamma}\right) \end{bmatrix}$$

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$$\begin{bmatrix} A_{22} & A_{2\Gamma} \\ A_{\Gamma 2} & A_{\Gamma\Gamma} + T^2_{1\to 2} \end{bmatrix} \begin{bmatrix} \vec{d}^3_2 \\ \vec{d}^3_{2\Gamma} \end{bmatrix}$$

$$= \langle \vec{w}^2_1, \vec{u}^1_{2\Gamma} - \vec{u}^0_{1\Gamma} \rangle \begin{bmatrix} \\ \vec{v}^2_1 \end{bmatrix}$$

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Normalize $\vec{d}^3_{2\Gamma}$ using α^3_2 such that $\vec{w}^3_2 = \alpha^3_2 \vec{d}^3_{2\Gamma}$ and calculate

$$\vec{v}^3_2 = \alpha^3_2 \left(-A_{\Gamma 2} \vec{d}^3_2 + T^1_{2\to 1} \vec{d}^3_{2\Gamma} \right)$$

Update $T^1_{2\to 1}$:

$$T^3_{2\to 1} = T^1_{2\to 1} - \vec{v}^3_2 \left(\vec{w}^3_2 \right)^\top$$

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Update $T_{i\rightarrow j}^n$ using $\vec{v}_i^n \left(\vec{w}_i^n\right)^\top$

WOODBURY MATRIX IDENTITY

The updates to $T^n_{1\to 2}$ and $T^n_{2\to 1}$ are low rank

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$$(A - VW^\top)\vec{u} = \vec{f},$$

$$\vec{u} = A^{-1}\vec{f} + A^{-1}V(I_{k\times k} - W^\top A^{-1}V)^{-1}W^\top A$$

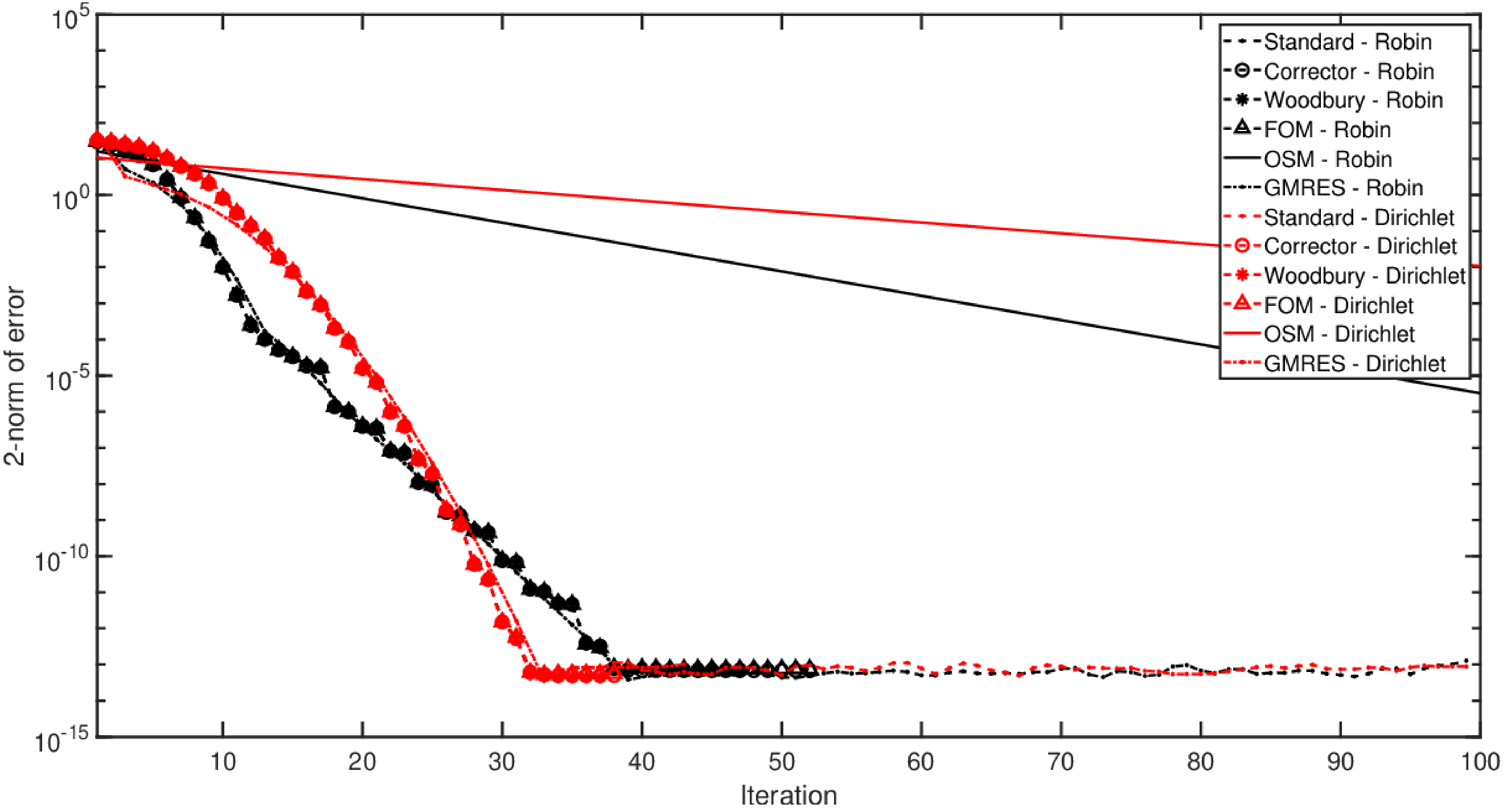
NUMERICAL EXPERIMENTS

Recall the sample problem:

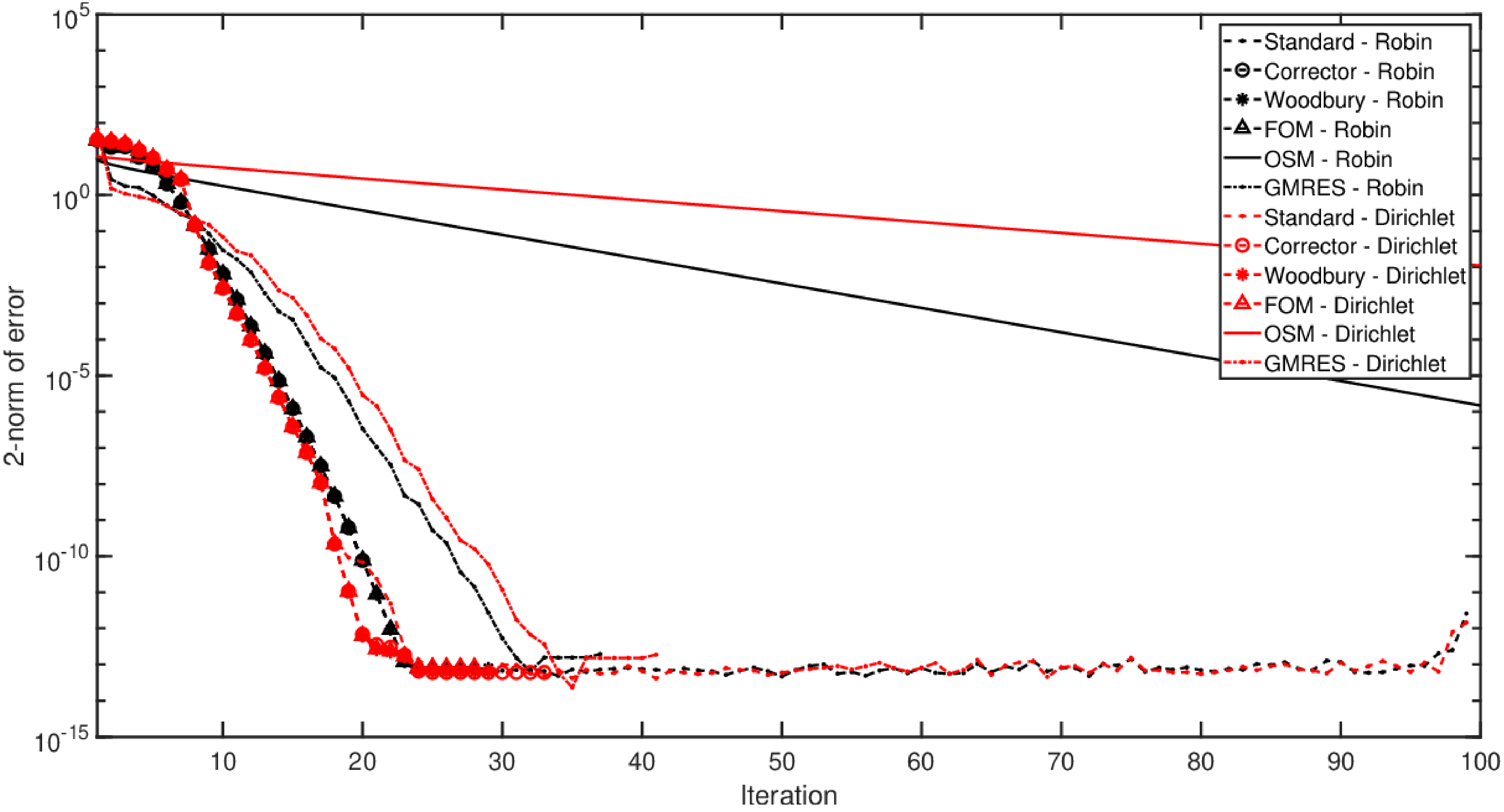
$$\Delta u(x, y) = f(x, y), \quad (x, y) \in \Omega = [-1, 1] \times [-1, 1]$$

$$u(x, y) = h(x, y), \quad (x, y) \in \partial\Omega$$

Let's apply AOSMs to this problem



Standard - Robin
Corrector - Robin
Woodbury - Robin
FOM - Robin
OSM - Robin
GMRES - Robin
Standard - Dirichlet
Corrector - Dirichlet
Woodbury - Dirichlet
FOM - Dirichlet
OSM - Dirichlet
GMRES - Dirichlet
2-norm of error
Iteration
10^{5}
10^{0}
10^{-5}
10^{-10}
10^{-15}
10
20
30
40
50
60
70
80
90
100



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FOM - Dirichlet
OSM - Dirichlet
GMRES - Dirichlet
2-norm of error
Iteration

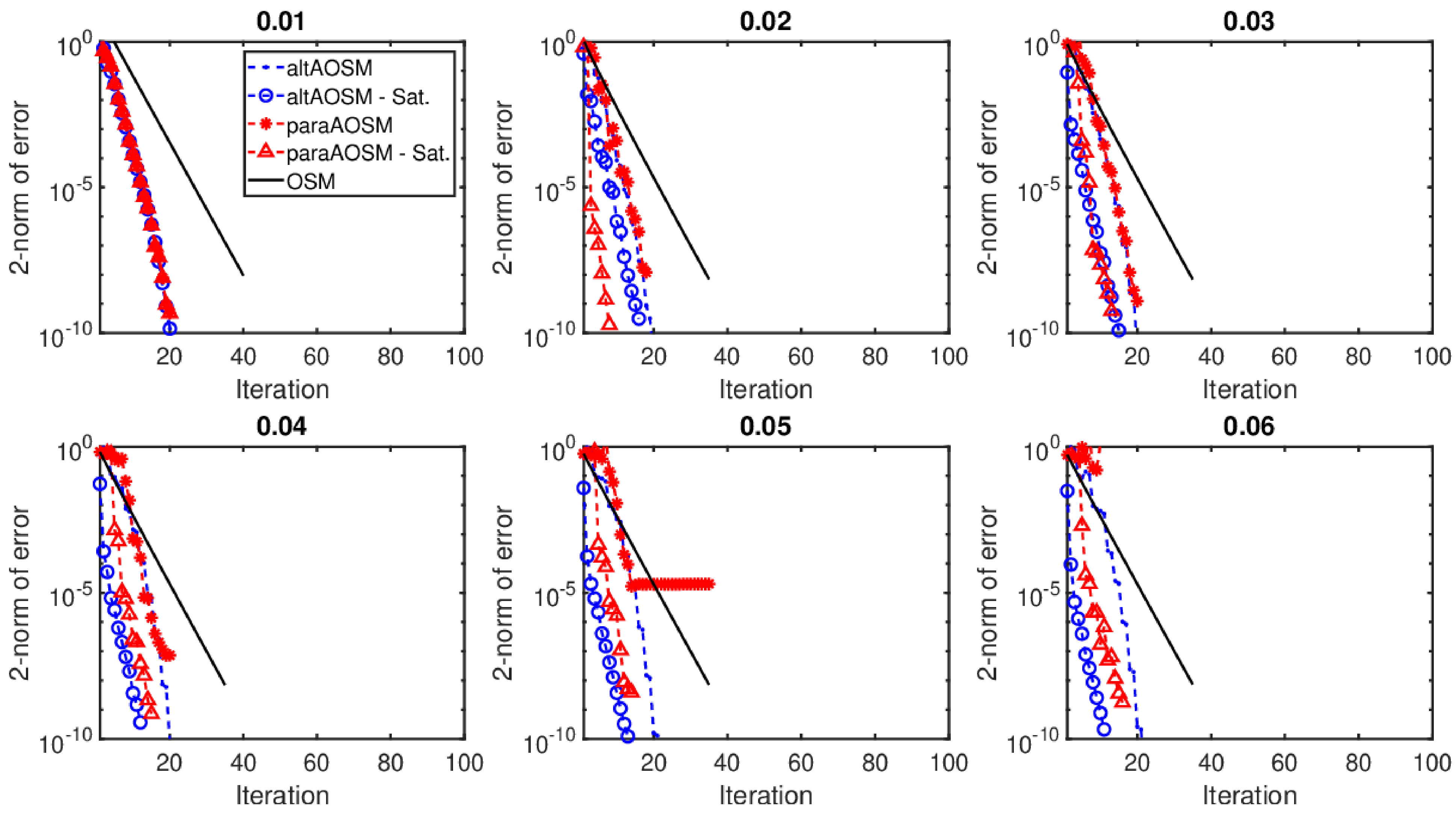
$$u_t(x,y,t) = \Delta u(x,y,t), \quad (x,y) \in \Omega = [-1,1] \times [-1,1],\ t \in [0,T]$$

$$u(x,y,0) = u_0(x,y), \quad (x,y) \in \Omega,$$

$$u(x,y,t) = h(x,y), \quad (x,y) \in \partial\Omega,\ t \in [0,T]$$

$$\frac{u_{n+1} - u_n}{\Delta t} = A u_{n+1}$$

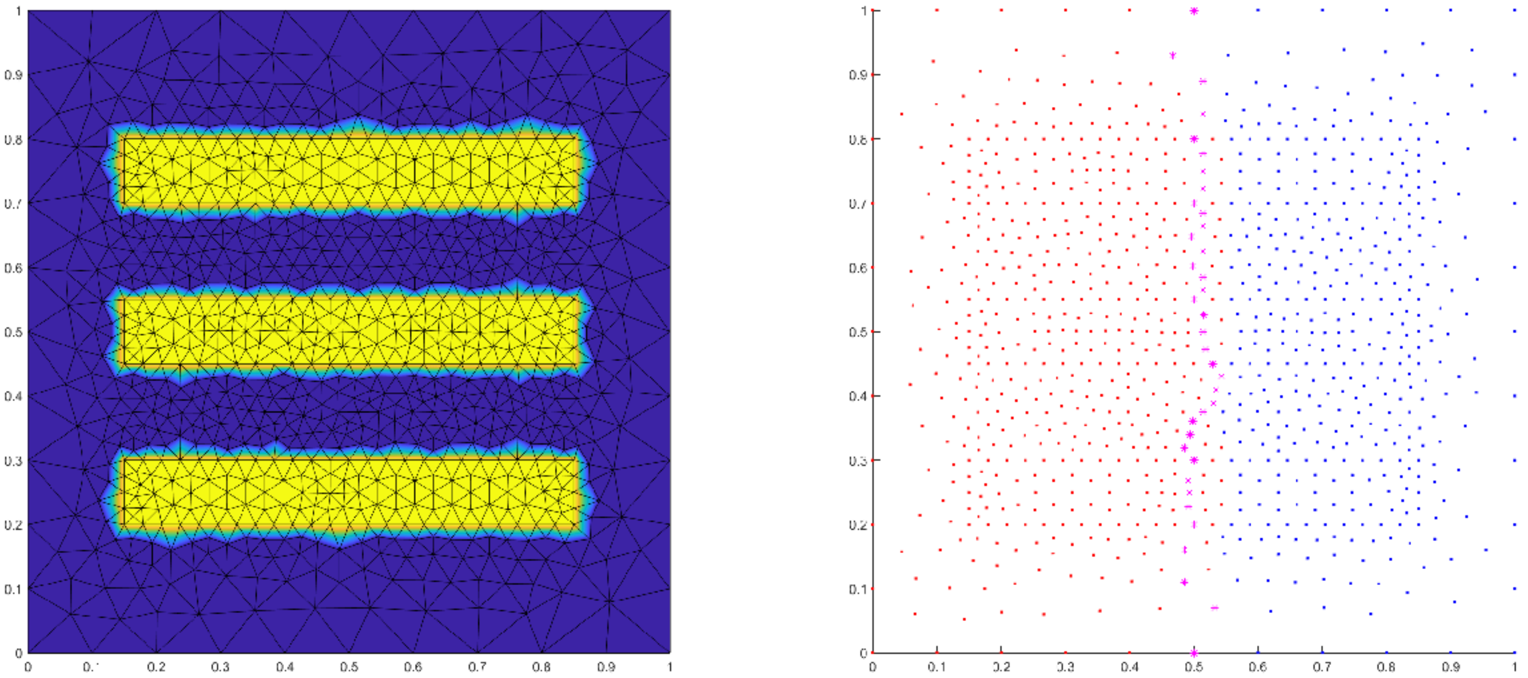
$$(I - \Delta t A) u_{n+1} = u_n$$


0.01
0.02
0.03
0.04
0.05
0.06
altAOSM
altAOSM - Sat.
paraAOSM
paraAOSM - Sat.
OSM
2-norm of error
Iteration
10^{0}
10^{-5}
10^{-10}
20
40
60
80
100

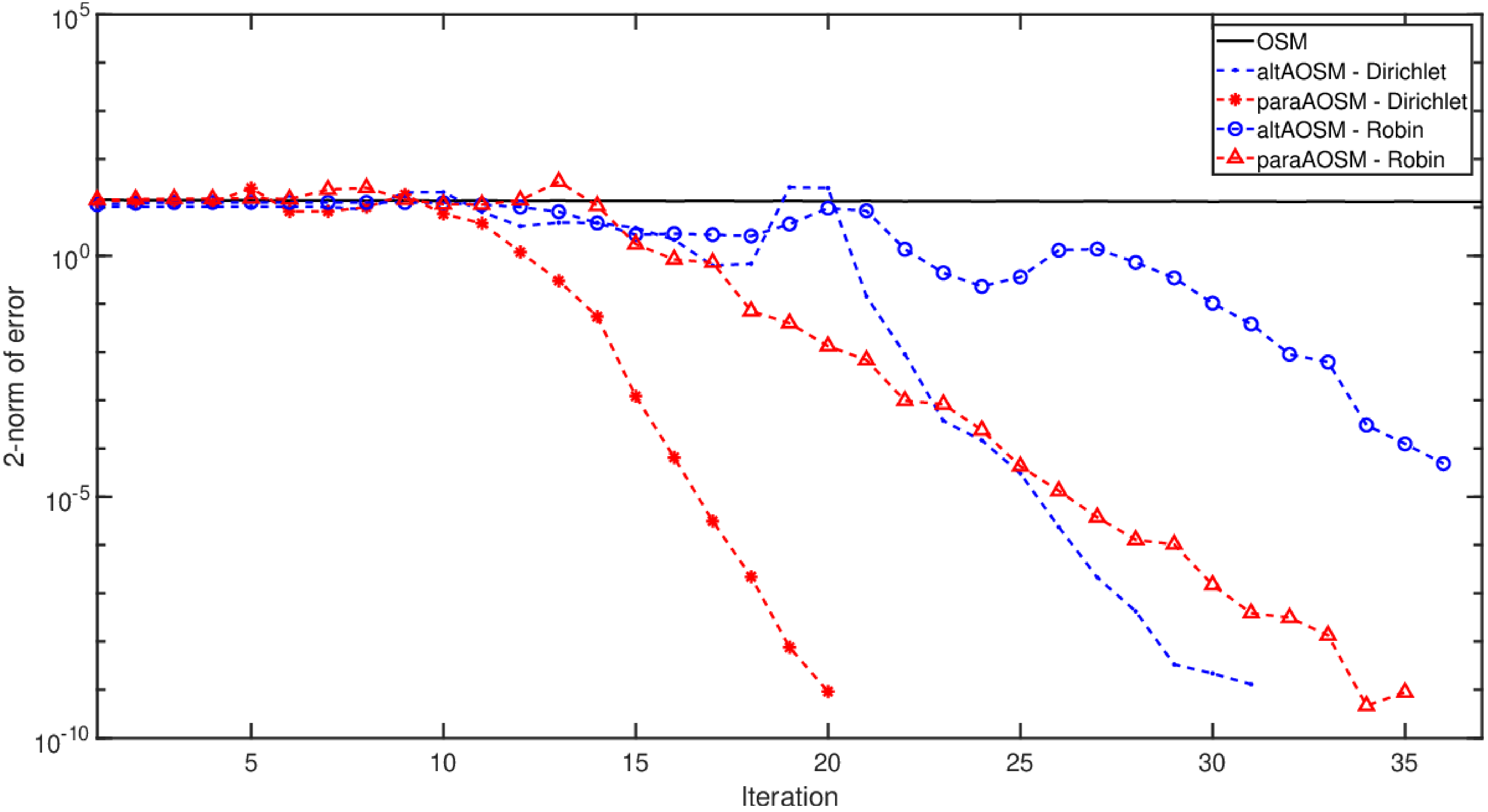
$$\nabla(\alpha(x,y)\cdot\nabla u(x,y)) = f(x,y), \quad (x,y)\in\Omega = [-1,1]\times[-1,1],$$

$$u(x,y) = h(x,y), \quad (x,y)\in\partial\Omega,$$

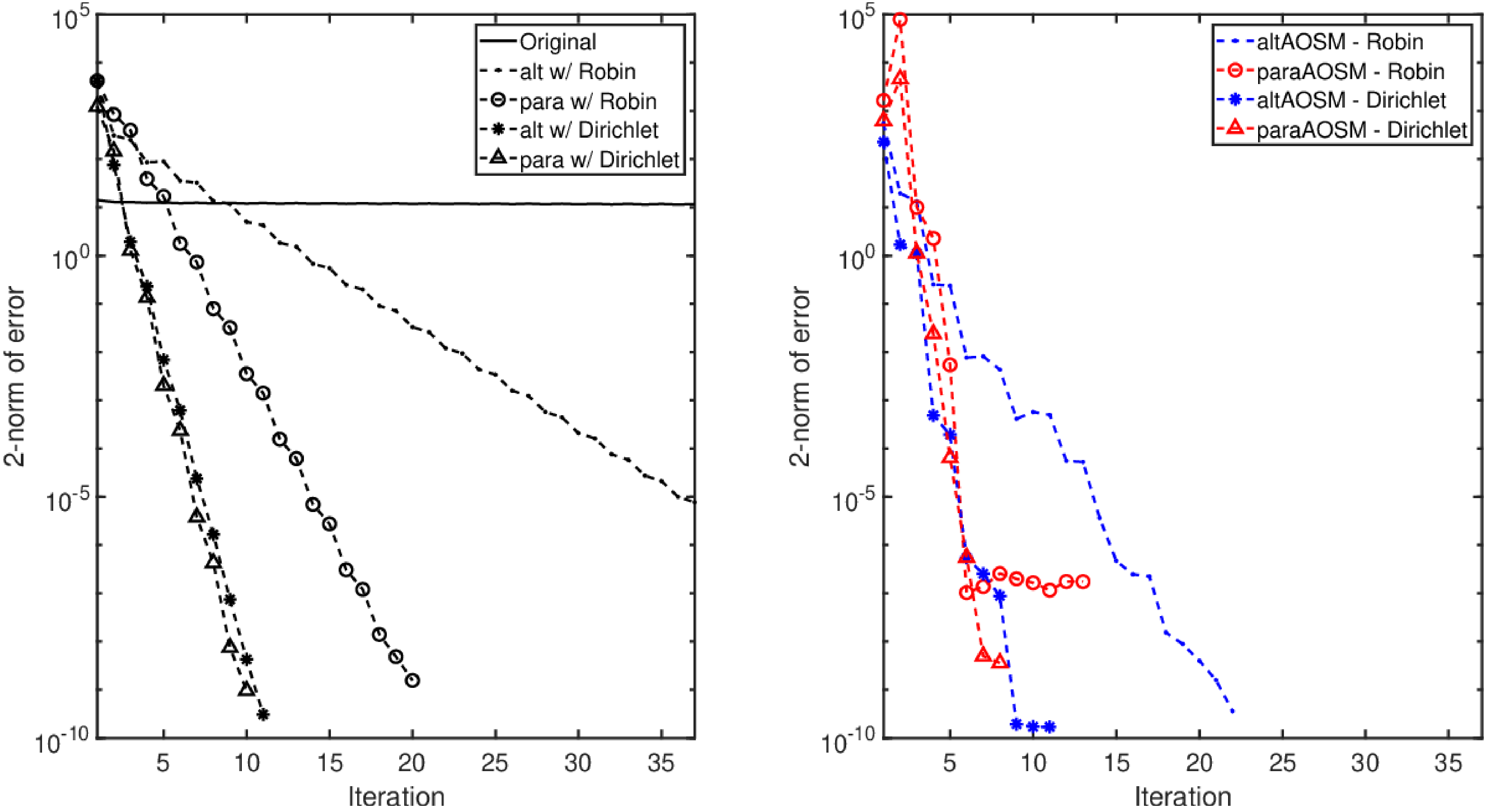
Solve this using FEM software



0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1



OSM
altAOSM - Dirichlet
paraAOSM - Dirichlet
altAOSM - Robin
paraAOSM - Robin
10^5
10^0
10^{-5}
10^{-10}
2-norm of error
5
10
15
20
25
30
35
Iteration



Original
alt w/ Robin
para w/ Robin
alt w/ Dirichlet
para w/ Dirichlet
2-norm of error
Iteration
altAOSM - Robin
paraAOSM - Robin
altAOSM - Dirichlet
paraAOSM - Dirichlet
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- Track down stability issues
- Test out other choices of adaptive transmission conditions